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Bell-Type Inequalities in Orthomodular Lattices. II. Inequalities of Higher Order

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This paper is a continuation of the first part and it is devoted to the study of Bell-type inequalities of order at least 3 in orthomodular lattices. We give some necessary and sufficient conditions for the validity of Bell-type inequalities of order 3 and also, more generally, for those of order n.

This paper is a continuation of Dvurečenskij and Länger (1995), hereafter referred to as [I]. Sections, theorems, and formulas are numbered in continuation of that work, starting with Section 8. References not listed in the present paper can be found at the end of [I].

8. BELL-TYPE INEQUALITIES OF ORDER 3

In this section, we investigate Bell-type inequalities of order 3. We show that these inequalities can entail a "Boolean" character of a given propositional system. Such a character has the inequality

$$p(a) + p(b) + p(c) - p(a \wedge b) - p(a \wedge c) - p(b \wedge c) \leq 1 \quad \text{for all} \quad a, b, c \in L \quad (8.1)$$

Let p be a state on an OML L. If there are a nonvoid set Ω, an algebra $\mathcal{S} \subseteq 2^{\Omega}$, a finitely additive probability measure P on $\mathcal{S}$, and a mapping[3] z:

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[3]For more details on $L_2(\Omega, S, P)$, where P is finitely additive, see Dunford and Schwartz (1957, Chapter III).

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$L \to L_2(\Omega, \mathcal{S}, P)$ such that, for all $a, b \in L$, (i) $E(z(a)z(b)) = p(a \wedge b)$ and (ii) $z(a \vee b) = z(a) + z(b)$ whenever $a \perp b$, then z is said to be a *random measure of the first kind* related to p.

Theorem 8.1. Let p be a state on an OML L, and let n be a positive integer. Then the following statements are equivalent:

(i) $p(\mathrm{com}(a, b)) = 1$ for all $a, b \in L$.

(ii) For p, (8.1) holds.

(iii) We have

$$p(a) + p(b) + p(c) - p(a \wedge b) - p(a \wedge c) - p(b \wedge c) + p(a \wedge b \wedge c) \leq 1 \qquad \text{for all} \quad a, b, c \in L \quad (8.2)$$

(iv) p is distributive.

(v) There are a Boolean algebra B, a homomorphism h from L onto B, and a (positive) state P on B such that $P(h(a)) = p(a)$ for any $a \in L$.

(vi) There is a random measure of the first kind related to p.

(vii) There exist real numbers $\alpha, \beta, \gamma, \delta$ with $0 < \beta \leq \alpha \leq 1$, $-\alpha - 1 \leq \gamma \leq -\alpha$, and $-1 - \alpha + \beta - \gamma \leq \delta \leq -\alpha + \beta - \gamma$ such that, for all $a, b \in L$, it holds that

$$1 - \alpha + \alpha p(a) + \alpha p(b) + \beta p(c) + \gamma p(a \wedge b) - \beta p(a \wedge c) - \beta p(b \wedge c) + \delta p(a \wedge b \wedge c) \leq 1$$

(viii) There exist real numbers $\alpha, \beta, \gamma, \delta$ with $-1 \leq \alpha \leq \beta < 0$, $-\alpha \leq \gamma \leq 1 - \alpha$, and $-\alpha + \beta - \gamma \leq \delta \leq 1 - \alpha + \beta - \gamma$ such that, for all $a, b \in L$, it holds that

$$0 \leq -\alpha + \alpha p(a) + \alpha p(b) + \beta p(c) + \gamma p(a \wedge b) - \beta p(a \wedge c) - \beta p(b \wedge c) + \delta p(a \wedge b \wedge c)$$

(ix) $p(t_1(a_1, \ldots, a_n)) = p(t_2(a_1, \ldots, a_n))$ holds for every $a_1, \ldots, a_n \in L$ and for every positive integer n, if $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$ is a law holding in any Boolean algebra.[4]

(x) For every $n \geq 1$ and for every $f: 2^{\{1,\ldots,n\}} \to \mathbb{R}$ ($f: 2^{\{1,\ldots,n\}} \to \mathbb{Z}$) with

$$\sum_{I \subseteq \{1,\ldots,n\}} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, \ldots, a_n \in \{0, 1\}$$

[4] By a term on an OML L we mean a mapping $t: L^n \to L$, where $t(x_1, \ldots, x_n)$ is an expression built up by the variables $x_1, \ldots, x_n \in L$ and the symbols (, $\vee$, $\wedge$,), $^\perp$, 0, 1. For example, if $t_1(x, y, z) = (x \vee y) \wedge z$ and $t_2(x, y, z) = (x \wedge z) \vee (y \wedge z)$ for all $x, y, z \in L$, then the equality $t_1 = t_2$ is a distributive law.

we have

$$\sum_{I \subseteq \{1,\ldots,n\}} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, \ldots, a_n \in L$$

(xi) For every $n \geq 1$ and for every $f\colon 2^{\{1,\ldots,n\}} \to \mathbb{R}$ ($f\colon 2^{\{1,\ldots,n\}} \to \mathbb{Z}$) with

$$\sum_{I \subseteq K} f(I) \in [0, 1] \qquad \text{for any} \quad K \subseteq \{1, \ldots, n\}$$

we have

$$\sum_{I \subseteq \{1,\ldots,n\}} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1] \qquad \text{for any} \quad a_1, \ldots, a_n \in L$$

(xii) $p(a) \leq p(a \wedge b) + p(a \wedge b^\perp)$ for all $a, b \in L$.

(xiii) $p(b) + p(c) - p(a \wedge b) - p(b \wedge c) - p(c \wedge d) + p(a \wedge d) \leq 1$ for all $a, b, c, d \in L$.[5]

(xiv) p is Jauch–Piron and if $a, b \in L$ and $p(a \wedge b) = p(a \wedge b^\perp) = 0$, then $p(a) = 0$.

Proof. It is evident that (v) $\Rightarrow$ (i), (iv), (ix), and each of these conditions implies (xii).

(xi) $\Rightarrow$ (ii), (iii), (vii), (viii). This is evident.

(ii), (iii), (vii), (viii) $\Rightarrow$ (xii). Put $b = a^\perp$.

(xii) $\Rightarrow$ (v). Calculate

$$\begin{aligned} p(a \vee b) &= p((a \vee b) \wedge a) + p((a \vee b) \wedge a^\perp) \\ &= p(a) + p((a \vee b) \wedge a^\perp \wedge b) + p((a \vee b) \wedge a^\perp \wedge b^\perp) \\ &= p(a) + p(b \wedge a^\perp) = p(a) + p(b) - p(a \wedge b) \end{aligned} \tag{8.3}$$

Hence, p is a valuation. Using the equivalence (iii) and (x) of Theorem 4.1, we can find a modular OML B, a homomorphism h from L onto B, and a positive subadditive state P on B such that $P(h(a)) = p(a)$, for any $a \in L$. We assert that B is a Boolean algebra. Indeed, for all $a, b \in L$, $\operatorname{com}(h(a), h(b)) = h(\operatorname{com}(a, b))$. In view of (i) [(xii) implies (i)], $P(\operatorname{com}(h(a), h(b))) = 1$, which entails that $\operatorname{com}(h(a), h(b)) = 1_B$.

(v) $\Rightarrow$ (vi). Due to the Stone representation theorem, we can assume that B is an algebra of subsets of a nonvoid set Ω, and that P is a state on B. Define a mapping $z\colon L \to L_2(\Omega, B, P)$ via $z(a) := \chi_{h(a)}$, $a \in L$. Then z is well defined and

$$E(z(a)z(b)) = E(\chi_{h(a)}\chi_{h(b)}) = P(h(a) \cap h(b)) = P(h(a \wedge b)) = p(a \wedge b)$$

[5] This is an inequality of Clauser–Horne type.

for all $a, b \in L$. If $a, b \in L$ and $a \perp b$, then $h(a) \cap h(b) = \emptyset$ and

$$z(a \vee b) = \chi_{h(a \vee b)} = \chi_{h(a) \cap h(b)} = \chi_{h(a)} + \chi_{h(b)} = z(a) + z(b)$$

(vi) $\Rightarrow$ (xii). $p(a) = E(z(a)z(b \vee b^{\perp})) = E(z(a)z(b)) + E(z(a)z(b^{\perp})) = p(a \wedge b) + p(a \wedge b^{\perp})$ for all $a, b \in L$.

(v) $\Rightarrow$ (xi). Follows from Theorem 3.4.

(x) $\Leftrightarrow$ (xi). Follows from Proposition 3.2.

(xiii) $\Leftrightarrow$ (i). Proved in Pulmannová (n.d.).

(xiv) $\Rightarrow$ (i). Let $a, b \in L$ and $c := \text{com}(a, b)$. Define $a_1 := a \wedge c^{\perp}$ and $b_1 := b \wedge c^{\perp}$. Then $a_1 \wedge b_1 = a_1 \wedge b_1^{\perp} = 0$, so that $p(a_1) = 0$. Similarly, $p(b_1) = 0$. Since $a_1 \vee b_1 = c^{\perp}$, the Jauch–Piron property of p entails $p(c^{\perp}) = 0$.

(v) $\Rightarrow$ (xiv). Straightforward. ■

From the last theorem we see that the any Bell-type inequality of order 3 holds practically on every OML which has a "Boolean character" for a state p. It is worth saying that if $L = L(H)$, dim $H \geq 3$, then not every Bell-type inequality of order 3 holds for every Gleason state on $L(H)$.

We recall that if any Bell-type inequality of order 3 holds for any state on an OML L, then it does not entail that L is a Boolean algebra. Indeed, modifying Example 4.2 and applying Lemma 6.5, we have the following example:

Example 8.2. Let L_0 be a stateless OML and B a Boolean algebra. Then any Bell-type inequality holds for any state on $L = L_0 \times B$, but L is not a Boolean algebra.

For any finite subset $M = \{a_1, \ldots, a_n\}$ of an OML L we define the *commutator*, com M, of M via

$$\text{com } M = \bigvee_{j_1, \ldots, j_n = 0}^{1} \bigwedge_{i=1}^{n} a_i^{j_i} \tag{8.4}$$

where $a^0 := a^{\perp}$, $a^1 := a$ for any $a \in L$. If $M = \emptyset$, we put com $M := 1$.

Theorem 8.3. Let $M = \{a_1, \ldots, a_n\}$ be a finite subset of an OML L and let p be a state on L. Any Bell-type inequality holds on the sub-OML $L_0(M)$ of L generated by M for $p | L_0(M)$ if and only if

$$p(\text{com } M) = 1 \tag{8.5}$$

Proof. Using (ix) of Theorem 8.1, we find that (8.5) follows easily. Conversely, let (8.5) hold. Define

$$J_0(M) := \{c \in L_0(M): c \leq (\text{com } M)^{\perp}\} \tag{8.6}$$

Then $J_0(M)$ is a p-ideal of $L_0(M)$ (Pulmannová, 1985).[6] The relation $\sim_M$ on $L_0(M)$ defined via $(a \vee b) \wedge (a \wedge b)^\perp \in J_0(M)$, $a, b \in L_0(M)$, is a congruence on $L_0(M)$, and, in addition, $B := L_0(M)/\sim_M$ is a Boolean algebra (Marsden, 1970). If h is the canonical homomorphism from $L_0(M)$ onto B, then the mapping P on B defined via $P(h(a)) := p(a)$, $a \in L_0(M)$, is, in view of (8.5), a state on B. Using again (v) and (xi) of Theorem 8.1, we have the assertion in question. ■

Theorem 8.4. Let M be a nonempty subset of an OML L and let p be a state on L. Any Bell-type inequality holds in the sub-OML $L_0(M)$ of L generated by M for $p \mid L_0(M)$ if and only if

$$p(\text{com } F) = 1$$

for any finite subset F on M.

Proof. This is the same as that of Theorem 8.3 (see also Dvurečenskij, 1993, Theorem 2.4.9); we only define

$$J_0(M) = \{c \in L_0(M): c \leq \bigvee_{i=1}^{n} (\text{com } F_i)^\perp,$$

$$F_i \subseteq M, |F_i| < \infty, 1 \leq i \leq n < \infty\} \quad ■$$

Theorem 8.5. Let $L = L(H)$ and p be a state on $L(H)$ of the form

$$p(M) = \sum_i \lambda_i \|P_M x_i\|^2, \qquad M \in L(H)$$

where $\lambda_i > 0$ for any i, $\Sigma_i \lambda_i = 1$, and $\{x_i\}$ is an orthonormal system of vectors in H. Let $M = \{M_1, \ldots, M_n\}$ be a finite set of closed subspaces of H. Then any Bell-type inequality holds in $L_0(M)$ for $p \mid L_0(M)$ if and only if

$$P_{M_1} \cdots P_{M_n} x_i = P_{M_{i_1}} \cdots P_{M_{i_n}} x_i$$

holds for any x_i and any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

Proof. Follows from Dvurečenskij (1993, Theorem 2.5.4) and Theorem 8.3. ■

Theorem 8.6. Let $M = \{a_1, \ldots, a_n\}$ be a finite subset of an OML L and let p be a state on L with support a_0. Any Bell-type inequality holds in the sub-OML $L_0(M)$ of L generated by M for $p \mid L_0(M)$ if and only if

$$a_0 \leq \text{com } M$$

[6] A nonempty subset J of an OML L is said to be a *p-ideal* of L if (i) $a \vee b \in J$ whenever $a, b \in J$, (ii) $a \in J$ whenever $a \in L$, $b \in J$, and $a \leq b$, (iii) $(a \vee b^\perp) \wedge b \in J$ whenever $a \in J$ and $b \in L$. It is possible to show that the relation $\sim_J$ on L, defined via $a \sim_J b$ iff $(a \vee b) \wedge (a \wedge b)^\perp \in J$ $(a, b \in L)$, is a congruence on L (Kalmbach, 1983).

Proof. Follows from the definition of the support and from Theorem 8.3. ■

Remark 8.7. Theorems 8.3 and 8.5 are of great importance for the study of classicality and nonclassicality of a given system of events. It can happen that, for example, $\{a_1, \ldots, a_n\}$ is a set of not pairwise commuting events, but $p(\text{com}\{a_1, \ldots, a_n\}) = 1$. Then $a_{i0} := a_i \wedge a$, $i = 1, \ldots, n$, where $a = \text{com}\{a_1, \ldots, a_n\}$, are mutually compatible events in the interval OML $L_{[0,a]} = \{b \in L: b \leq a\}$, and $p | L_{[0,a]}$ is a state on $L_{[0,a]}$. Therefore, any statistical information involved in $\{a_1, \ldots, a_n\}$ remains the same for $\{a_{10}, \ldots, a_{n0}\}$ in $L_{[0,a]}$, and $L_{[0,a]}$ can serve as a classical probability model.

Remark 8.8. The commutator of any subset M of an OML L is defined via

$$\text{com}\, M := \bigwedge \{\text{com}\, F: F \subseteq M, |F| < \infty\} \tag{8.7}$$

supposing that (8.7) exists in L. Varying the example of Poguntke (1980; see also Dvurečenskij, 1993, Example 2.4.20), we have the following result: Let $L_1 = \{0, 1\}$ and $L_2 = \text{MO2}$. Let $L_0 := L_1^{\aleph_0} \times L_2^{\aleph_0}$, and let L be the sub-OML of L_0 generated by all elements $(\{a_n\}_n, \{b_n\}_n)$, where either $\{n: a_n \neq 0\} \cup \{m: b_m \neq 0\}$ is finite or $\{n: a_n \neq 1\} \cup \{m: b_m \neq 1\}$ is finite. If F is a finite subset of L, then $\text{com}\, F = (\{1\}_n, \{b_n\}_n)$, where b_n is either 0 or 1. Therefore, com L does not exist in L.

On the other hand, on L_1 there is a unique state p_1, namely $p_1(0) = 0$, $p_1(1) = 1$. If K is a finite subset of $\{1, 2, \ldots\}$, define a state p on L as follows: $p((\{a_n\}, \{b_n\}_n)) := \Sigma\, p_1(a_n)/|K|$ for any $(\{a_n\}_n, \{b_n\}_n) \in L$. Then $p(\text{com}\, F) = 1$ for any finite subset F of L, although com L does not exist in L. Therefore, the condition in Theorem 8.4 cannot be changed automatically to $p(\text{com}\, M) = 1$.

9. BELL-TYPE INEQUALITIES OF ORDER n

In the present section we shall deal with general Bell-type inequalities of order n. We recall that $N := \{1, \ldots, n\}$.

Theorem 9.1. Let p be a state on an OML L, f: $2^N \to \mathbb{R}$, and assume that (i) or (ii) holds:

(i) We have

$$\sum_{I \subseteq N} f(I)p\left(\bigwedge_{i \in I} a_i\right) \leq 1 \qquad \text{for all} \quad a_1, \ldots, a_n \in L$$

and there exist $j, k \in N$, $j \neq k$, with $f(\{j\}) = f(\{k\}) = -f(\{j, k\}) = 1 - f(\emptyset) > 0$.

(ii) We have

$$0 \leq \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \qquad \text{for all} \quad a_1, \ldots, a_n \in L$$

and there exist $j, k \in N$, $j \neq k$, with $f(\{j\}) = f(\{k\}) = -f(\{j, k\}) = -f(\varnothing) < 0$.

Then p is subadditive.

Proof. If (i) holds, then consider the inequality $\Sigma_{I \subseteq N} f(I) p(\wedge_{i \in I} a_i) \leq 1$ for $a_1, \ldots, a_n \in L$, with $a_i = 0$ for all $i \in N \backslash \{j, k\}$ and apply Theorem 4.1. The case that (ii) holds is treated in a completely analogous way. ■

Theorem 9.2. Let p be a state on an OML L, f: $2^N \to \mathbb{R}$, and assume that (i) or (ii) holds:

(i) We have

$$\sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \leq 1 \qquad \text{for all} \quad a_1, \ldots, a_n \in L$$

and there exist $j, k, m \in N$, $j \neq k \neq m \neq j$, with $f(\{j\}) = f(\{k\}) = 1 - f(\varnothing)$, and $f(\{m\}) = -f(\{j, m\}) = -f(\{k, m\}) > 0$.

(ii) We have

$$0 \leq \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \qquad \text{for all} \quad a_1, \ldots, a_n \in L$$

and there exist $j, k, m \in N$, $j \neq k \neq m \neq j$ with $f(\{j\}) = f(\{k\}) = -f(\varnothing)$, and $f(\{m\}) = -f(\{j, m\}) = -f(\{k, m\}) < 0$.

Then p is distributive.

Proof. Consider the case $a_i = 0$ for all $i \in N \backslash \{j, k, m\}$, $a_k = a_j^{\perp}$, and apply Theorem 8.1. ■

Theorem 9.3. Let p be a state on an OML L, $a_1, \ldots, a_n \in L$, and assume that

$$0 \leq \sum_{K \subseteq I \subseteq N} (-1)^{|I \backslash K|} p\left(\bigwedge_{i \in I} a_i\right) \qquad \text{for all} \quad K \subseteq N \tag{9.1}$$

Then

$$\sum_{K \subseteq I \subseteq N} (-1)^{|I \backslash K|} p\left(\bigwedge_{i \in I} a_i\right) \leq 1 \qquad \text{for all} \quad K \subseteq N \tag{9.2}$$

and (3.1) holds for all f: $2^N \to \mathbb{R}$ satisfying (3.2).

Proof. Calculate

$$\sum_{K \subseteq N} \sum_{K \subseteq I \subseteq N} (-1)^{|I \setminus K|} p\left(\bigwedge_{i \in I} a_i\right) = \sum_{I \subseteq N} p\left(\bigwedge_{i \in I} a_i\right) \sum_{K \subseteq I} (-1)^{|I \setminus K|}$$

$$\sum_{I \subseteq N} p\left(\bigwedge_{i \in I} a_i\right) \sum_{j=0}^{|I|} \binom{|I|}{j} (-1)^j = \sum_{I \subseteq N} p\left(\bigwedge_{i \in I} a_i\right) \delta_{\emptyset I} = 1$$

and

$$\sum_{K \subseteq N} \sum_{K \subseteq I \subseteq N} (-1)^{|I \setminus K|} p\left(\bigwedge_{i \in I} a_i\right) \sum_{J \subseteq K} f(J)$$

$$= \sum_{J \subseteq N} f(J) \sum_{I \subseteq N} p\left(\bigwedge_{i \in I} a_i\right) \sum_{J \subseteq K \subseteq I} (-1)^{|I \setminus K|}$$

$$= \sum_{J \subseteq N} f(J) \sum_{J \subseteq I \subseteq N} p\left(\bigwedge_{i \in I} a_i\right) \sum_{j=0}^{|I \setminus J|} \binom{|I \setminus J|}{j} (-1)^j$$

$$= \sum_{J \subseteq N} f(J) \sum_{J \subseteq I \subseteq N} p\left(\bigwedge_{i \in I} a_i\right) \delta_{IJ} = \sum_{J \subseteq N} f(J) p\left(\bigwedge_{i \in J} a_i\right) \quad \blacksquare$$

Remark 9.4. (i) The Bell-type inequality (9.1) satisfies (3.2):

$$\sum_{J \subseteq I \subseteq K} (-1)^{|I \setminus J|} = 0$$

if $J, K \subseteq N$ and $J \not\subseteq K$, and

$$\sum_{J \subseteq I \subseteq K} (-1)^{|I \setminus J|} = \sum_{j=0}^{|K \setminus J|} \binom{|K \setminus J|}{j} (-1)^j = \delta_{JK}$$

if $J \subseteq K \subseteq N$.

(ii) If $K \subseteq N$ and $|K| \geq n - 1$, then

$$0 \leq \sum_{K \subseteq I \subseteq N} (-1)^{|I \setminus K|} p\left(\bigwedge_{i \in I} a_i\right) \leq 1$$

for any state p on L and all $a_1, \ldots, a_n \in L$.

(iii) A state p on L is subadditive iff

$$0 \leq \sum_{I \subseteq \{1,2\}} (-1)^{|I|} p\left(\bigwedge_{i \in I} a_i\right) \qquad \text{for all} \quad a_1, a_2 \in L$$

(see Theorem 4.1).

(iv) A state p on L is distributive iff

$$0 \leq \sum_{I \subseteq \{1,2,3\}} (-1)^{|I|} p\left(\bigwedge_{i \in I} a_i\right) \qquad \text{for all} \quad a_1, a_2, a_3 \in L$$

(see Theorem 8.1).

Theorem 9.5. Let p be a state on an OML L. The following assertions are equivalent:

(i) We have

$$0 \leq \sum_{K \subseteq I \subseteq N} (-1)^{|I \setminus K|} p\left(\bigwedge_{i \in I} a_i\right)$$

for all $K \subseteq N$ and for all $a_1, \ldots, a_n \in L$.

(ii) We have

$$0 \leq \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \leq 1$$

for all $a_1, \ldots, a_n \in L$ and for all f: $2^N \to \mathbb{R}$ with (3.2).

Proof. Use Theorem 9.3 and (i) of Remark 9.4. ■

Theorem 9.6. Let p be a state on L, $a_1, \ldots, a_n \in L$, and f: $2^N \to \mathbb{R}$ such that

$$\sum_{I \in \mathcal{M}} f(I), \qquad \sum_{I \subseteq N} f(I) \in [0, 1]$$

for all $\mathcal{M} \subseteq 2^N$ with $\varnothing \in \mathcal{M}$ and $N \notin \mathcal{M}$. Then

$$0 \leq \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \leq 1$$

Proof. Put $a := \wedge_{i=1}^{n} a_i$. Since $p(a) \leq p(\wedge_{i \in I} a_i) \leq 1$ for all $I \subseteq N$, we have

$$\beta := f(\varnothing) + \sum_{\substack{\varnothing \neq I \subsetneq N \\ f(I) < 0}} f(I) + \left(\sum_{\substack{\varnothing \neq I \subsetneq N \\ f(I) > 0}} f(I) + F(N)\right) p(a)$$

$$\leq \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right)$$

$$\leq f(\varnothing) + \sum_{\substack{\varnothing \neq I \subsetneq N \\ f(I) > 0}} f(I) + \left(\sum_{\substack{\varnothing \neq I \subsetneq N \\ f(I) < 0}} f(I) + f(N)\right) p(a) =: \gamma$$

In view of $0 \le p(a) \le 1$, we have that β lies between

$$f(\varnothing) + \sum_{\substack{\varnothing \neq I \subseteq N \\ f(I) < 0}} f(I) \qquad \text{and} \qquad \sum_{I \subseteq N} f(I)$$

and γ lies between

$$f(\varnothing) + \sum_{\substack{\varnothing \neq I \subseteq N \\ f(I) > 0}} f(I) \qquad \text{and} \qquad \sum_{I \subseteq N} f(I)$$

According to our assumptions, all four sums lie between 0 and 1, which completes the proof. ■

Remark 9.7. (i) If p is a state on L, $a_1, \ldots, a_n \in L$, and $f\colon 2^N \to [0, 2^{-n}]$, then

$$0 \le \sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \le 1$$

(ii) Let p be a state on an OML L having two elements a, b such that $p(a) = p(b) = 1$ and $p(a \wedge b) = 0$. If $f\colon 2^{\{1,2\}} \to \mathbb{R}$ such that $0 \le \Sigma_{I \subseteq \{1,2\}} f(I) p(\wedge_{i \in I} a_i) \le 1$, for all $a_1, a_2 \in L$, then $\Sigma_{I \in \mathcal{M}} f(I) \in [0, 1]$ for all $\mathcal{M} \subseteq 2^{\{1,2\}}$ with $\varnothing \in \mathcal{M}$ and $N \notin \mathcal{M}$. (Indeed, we have $f(\varnothing) + f(\{1\}) + f(\{2\}) = f(\varnothing) + f(\{1\})p(a) + f(\{2\})p(b) + f(\{1, 2\})p(a \wedge b)$.)

Theorem 9.8. For every $n > 2$, the subadditivity of a state p on an OML L is equivalent to the fact that

$$\begin{aligned} 0 \le p(a_2) + \cdots + p(a_{n-1}) - p(a_1 \wedge a_2) - p(a_2 \wedge a_3) \\ - \cdots - p(a_{n-1} \wedge a_n) + p(a_n \wedge a_1) \end{aligned}$$

for all $a_1, \ldots, a_n \in L$.

Proof. According to Theorem 4.1, the subadditivity of p is equivalent to the fact that $S_p(a, c) \le S_p(a, b) + S_p(b, c)$ for all $a, b, c \in L$. Since $S_p(a, a) = 0$ for all $a \in L$, the validity of the triangle inequality for S_p is, for every fixed $n > 2$, equivalent to the assertion that

$$S_p(a_1, a_n) \le S_p(a_1, a_2) + S_p(a_2, a_3) + \cdots + S_p(a_{n-1}, a_n)$$

for all $a_1, \ldots, a_n \in L$. ■

Theorem 9.9. Let $L_1, \ldots, L_m$ be OMLs. Put $L := L_1 \times \cdots \times L_m$ and let $f\colon 2^N \to \mathbb{R}$. Then the following assertions are equivalent:

(i) We have

$$\sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1]$$

for all states p on L and all $a_1, \ldots, a_n \in L$.

(ii) We have

$$\sum_{I \subseteq N} f(I) p\left(\bigwedge_{i \in I} a_i\right) \in [0, 1]$$

for all $j = 1, \ldots, m$, all states p on L_j, and all $a_1, \ldots, a_n \in L_j$.

Proof. Since the states p on L are exactly the mappings from L to $[0, 1]$ of the form $p((b_1, \ldots, b_m)) = \Sigma_{j \in J} \alpha_j p_j(b_j)$ for all $(b_1, \ldots, b_m) \in L$, where $\emptyset \neq J \subseteq \{1, \ldots, m\}$, $\alpha_j > 0$ for all $j \in J$, $\Sigma_{j \in J} \alpha_j = 1$, and p_j is a state on L_j for every $j \in J$, then (i) is equivalent to the assertion that

$$S := \sum_{I \subseteq N} f(I) \sum_{j \in J} \alpha_j p_j\left(\bigwedge_{i \in I} a_i\right) \in [0, 1]$$

for all nonempty subsets J of $\{1, \ldots, m\}$, all families $\alpha_j, j \in J$, of positive reals with $\Sigma_{j \in J} \alpha_j = 1$, all families $p_j, j \in J$, of states on L_j, and all $a_1, \ldots, a_n \in L_j$. Since

$$S = \sum_{j \in J} \alpha_j \sum_{I \subseteq N} f(I) p_j\left(\bigwedge_{i \in I} a_i\right)$$

the latter assertion is equivalent to (ii). ∎

10. CONCLUDING REMARKS

In the first part of this work we showed that if a Bell-type inequality of order n holds for a certain state in an orthomodular lattice, then it holds in any classical case (Proposition 3.1); the converse implication does not hold, in general. We have studied the connection between the validity of the original Bell inequality of order 2 in orthomodular lattices and different properties of the corresponding state. The criteria for the validity of this inequality are presented in Theorem 4.1.

The validity of Bell-type inequalities of order 2 is studied (i) in the most important quantum logic $L(H)$, the system of all closed subspaces of a Hilbert space H (Proposition 5.2), (ii) in $\mathcal{P}_A(H)$, the system of all skew projections on H (Proposition 5.5), and (iii) in Krein spaces (Example 6.2).

The validity of the original Bell inequality for a family of states may entail the distributivity of the corresponding orthomodular lattice (Theorems 7.3–7.6).

In the second part of this work we first presented criteria for the validity of Bell-type inequalities of order 3. They imply the distributive character of L with respect to the corresponding state (Theorem 8.1). Theorem 8.4 provides a criterion for the validity of Bell-type inequalities of order 3 in L by means of certain conditions on a generating set of L. The general discussion on Bell-type inequalities of order n is presented in Section 9.

Finally, the following results should be pointed out:

(i) For every Boolean algebra and for every state on it all Bell-type inequalities are valid (Section 3).

(ii) This property does not characterize the class of Boolean algebras. This means that there exist OMLs L with a nonempty state space which have the property that all Bell-type inequalities hold for all states on L, but which are not Boolean algebras (Example 8.2). All Bell-type inequalities are valid for a state p on an OML L iff L is distributive with respect to p (i.e., if p is distributive) (Theorem 8.1).

(iii) The original Bell inequality implies all possible Bell-type inequalities of order 2 (Theorem 4.1).

(iv) There exist OMLs L and states p on L such that all Bell-type inequalities of order 2 are valid, but not all Bell-type inequalities of order 3 hold (Proposition 5.2).

(v) There exist single Bell-type inequalities of order 3 (Theorem 8.1) and also of higher order (Theorem 9.2) which imply all possible Bell-type inequalities.

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